Randomness and determinism in soap froth dynamics

Boris Levitan
Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel
(Received 20 June 1996)

A topological model of soap froth evolution with deterministic topological rearrangements ($T2$ processes) agrees with experiments, in contrast with a model that performs $T2$ processes at random. However, in some physical situations the $T2$ processes are not purely deterministic. The present work analyzes the interplay between randomness and determinism in $T2$ processes by simulating a topological model, in which the randomness is controlled by a parameter. The results agree with the suggested mean-field description. A possible experimental test of these results is discussed. [S1063-651X(97)09101-0]

PACS number(s): 82.70.Rr, 02.50.Fz, 05.70.Ln

Soap froth, confined between two closely spaced parallel plates, forms a cellular structure whose time evolution exhibits a number of interesting features [1]. The starting point for the study of the froth's dynamics is von Neumann's law, which states how the area a_n of any n-sided bubble changes in time [2]:

$$\frac{da_n}{dt}=k(n-6), \tag{1}$$

where k absorbs all the material constants.

As Eq. (1) implies, bubbles with $n<6$ sides shrink and disappear, so the total number of bubbles in the system decreases. Since no new bubbles appear and the total area of the sample is fixed, the mean area of a bubble $\overline{a}$ increases. After a transient period [3,4], which depends on the initial configuration, coarsening approaches the *universal* scaling regime [5] characterized by linear growth of the mean area $\overline{a}\sim t$ and steady distributions of dimensionless areas (in units of $\overline{a}$) and the number of sides of the bubbles.

When a bubble disappears, its neighbors can lose or gain sides [6]. Such sudden *topological rearrangements* are called $T2$ processes. While Eq. (1) determines the evolution of the froth in terms of a_n and n in a *unique* way, the outcome of a $T2$ process is *not* uniquely determined by these variables (except for triangles): there are *two* possibilities for the decay of a rectangle and *five* for a pentagon. The correct outcome can be determined *only* by specifying the total dynamics of the mechanical equilibration [7]. Only by solving such dynamics can one simulate the *deterministic* evolution of a given initial configuration [8].

One can avoid this complicated problem using a so-called topological approach [9]. Select the $T2$ outcome at random [10,11] or apply any other plausible *rule* that defines the $T2$ processes in terms of a_n and n only [12,13]. Then the froth's dynamics is treated as follows. The bubbles' areas evolve according to Eq. (1). Whenever a bubble's area approaches zero, a $T2$ process is performed according to the chosen rule. To perform $T2$ processes one needs to keep only the *topology* (a list of neighbor) and *not* the explicit spatial configuration of the froth. The topological information can be stored in the *adjacency* matrix [10]. Then the $T2$ processes are equivalent to certain transformations of this matrix.

The model based on random $T2$ processes gives satisfactory results for the distribution of the number of sides (topological distribution) in the scaling state [10,11], but disagrees with experiments in a *qualitative* way for a number of more complicated problems [13]. We will consider here one of them, the evolution of a single defect [14] in an ideal network of hexagonal bubbles (the defect is created by a $T1$ switch of a single side). Were it not for the defect, the system would have been steady [see Eq. (1)]. So the network consists of two parts: the evolving neighborhood of the defect (the "perturbed" part) and the "inert" hexagons that do not evolve. A propagating front separates them.

Simulations of this problem by the random model [15] predicted that the topological distribution of the growing cluster approaches a steady form with second moment $\mu_2\approx 0.8$. More detailed numerical experiments [16,17], where the effects of the spatial configuration of the froth are taken into account explicitly, show, however, the *unbounded* growth of μ_2. The cluster consists of one very large bubble (with its number of sides being of the order of the number of bubbles in the cluster) surrounded mostly by five-, six-, and seven-sided bubbles, resulting in the unbounded growth of μ_2 in a straightforward way.

This discrepancy originates from the nonrandomness of the $T2$ processes. Indeed, the more realistic topological model (model C in Ref. [13]), which performs $T2$ processes *deterministically*, reproduces the cluster's structure described above. The explicit rules are not essential for us now; we note only that in model C a many-sided bubble more readily receives a side when a pentagon vanishes and more reluctantly loses a side when a rectangle vanishes, compared to the random $T2$ case, promoting the growth of the largest bubbles.

This failure of the random model does not mean that it is totally irrelevant. Indeed, one can imagine situations where the $T2$ processes look random. When a four- or five-sided bubble shrinks to a point it turns into a *fourfold* or *fivefold* vertex that is *mechanically unstable* and decays into one of the possible stable configurations of threefold vertices. Generally, the result of the decay of an unstable state into one of several stable states may depend on very tiny details of the real system. In this sense, the deterministic model relates to the "ideal" situation, which can be perturbed, for example, by roughening the plates' surfaces, especially if the distance

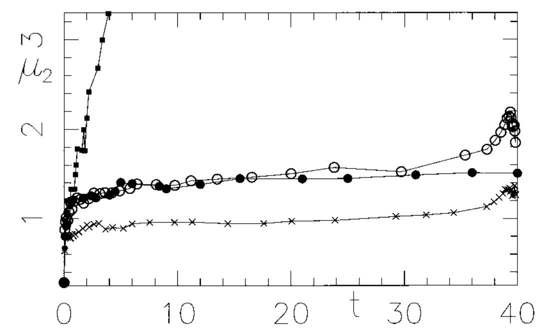

FIG. 1. Second moment of the topological distribution of the cluster μ_2 as a function of the number of killed bubbles t obtained by simulations on a lattice of 40 000 bubbles for three values of the noise parameter β: $\beta=0.1$ (solid squares), $\beta=0.7$ (crosses), $\beta=0.2$ (open circles). Peaks of these curves at the very end of the evolution are due to finite-size effects, as seen from the comparison with larger simulations: $\beta=0.2$ (bullets) for 100 000 bubbles lattice.

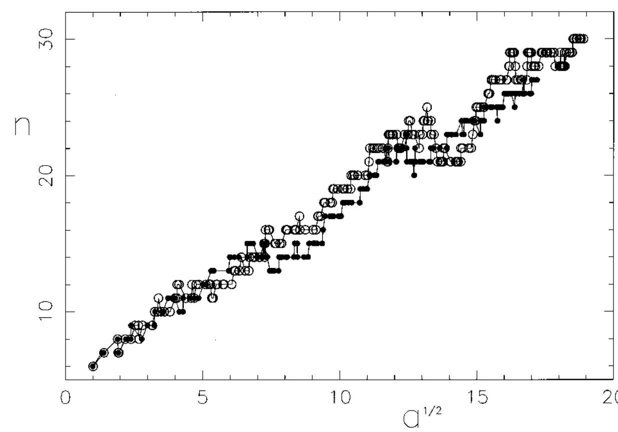

FIG. 2. Typical evolution of large bubbles. Dependence of the number of sides n on $\sqrt{a}$ (a is the area of the bubble) for two bubbles in the same cluster, at $\beta=0.17$.

between the plates D is much greater than the soap walls' width d as suggested in Ref. [7].

When $D \gg d$, the irregularities of the plates' shape are magnified by the three-dimensional capillary instability of the vanishing bubble and therefore can effectively randomize the $T2$ processes. In this case, the random approach seems relevant. In the opposite limit $D \ll d$, the situation is quite stable and $T2$ processes are deterministic [7]. Accordingly, the long-time evolution of the single cluster is qualitatively different in these two limits, answering the question posed at the end of Ref. [7], whether the difference between the two limits leads to dissimilar macroscopic behavior at long times.

In the present work we ''bridge'' these two extremes, introducing noise into the deterministic model (model C of Ref. [13]). Experimentally, the noise is governed by the plates' roughness and by the D/d ratio; in the Potts model simulation, like that of Ref. [16], the $T2$ processes can be randomized by the temperature as well as by the quenched disorder.

Thus we introduce the noise parameter β $(0<\beta<1)$, the probability that a given $T2$ process will happen at random. At the two extremes we have the deterministic ($\beta=0$) and random ($\beta=1$) approaches, so that μ_2 growth is unbounded for $\beta=0$, while for $\beta=1$ it stays finite. Therefore, gradually increasing β from 0 to 1, one expects a *transition* from one type of behavior to another at some $\beta=\beta_c$. On the other hand, one cannot *a priori* rule out the possibility that for any $\beta\neq 0$, $\mu_2(t)$ is bounded and converges to its limit $\mu_2(\beta)$ and $\mu_2(\beta)\to\infty$ when $\beta\to 0$.

I have simulated the model for different β on a network of 40 000 bubbles with periodic boundary conditions. At small β one of the bubbles rapidly increases its number of sides n and in a short time achieves $n=30$, when the simulations automatically stopped. For large β such large bubbles are not observed. Typical dependences $\mu_2(t)$ for three values of β are presented in Fig. 1. For $\beta=0.2$ and $\beta=0.7$, μ_2 achieves its equilibrium value in a very short time [18]. On the other hand, for $\beta=0.1$, μ_2 grows linearly in time.

Increasing β, the behavior changes at $\beta=0.17$. I have performed extensive simulations for $\beta=0.17$ on a network of 100 000 bubbles. The maximum number of bubbles in the cluster was $N=4250$. In some runs, the topological distribution of the cluster rapidly approached a steady form (see Table I) with $\mu_2\approx 1.65$ (when $N\sim 300$) and then stayed almost unchanged until the end (''large β'' behavior). In other runs, a large bubble appeared and grew until $n=30$ (''small β'' behavior). A typical evolution of such bubbles is presented in Fig. 2. In fact, this coexistence has been observed not only at $\beta=0.17$ but in the whole interval $0.17<\beta<0.20$; the ''large β'' behavior did not occur for $\beta<0.17$ and the ''small β'' behavior did not occur for $\beta>0.20$. This coexistence is due to the finite duration of the simulations and is related to the ''nucleation time'' as discussed below.

Let us turn to the mean-field description. We define the probabilities for a given n-sided bubble to gain and to lose a side in a $T2$ process: $P_{\text{gain}}=c_+n$, $P_{\text{lose}}=c_-n$ (the coefficients c_+ and c_- are the corresponding probabilities per side). For random $T2$ processes, the following expressions were obtained in a mean-field approximation [19]:

$$c_-^{\text{ran}}=3w_3x_3+2w_4x_4+2w_5x_5, \quad c_+^{\text{ran}}=w_5x_5. \tag{2}$$

TABLE I. Topological distribution of the cluster obtained from simulations at $\beta=0.17$. The data are averaged over the later stage of evolution (the cluster contained 2000–3000 bubbles). No single large bubble appeared in this run.

n	3	4	5	6	7	8	9
x_n	0.0018	0.046	0.26	0.52	0.11	0.040	0.011
n	10	11	12	13	14	15	16
$100x_n$	0.78	0.38	0.23	0.075	0.075	0.050	0.025

Here x_n are the relative fractions of the n-sided bubbles; w_n are their relative disappearance rates. Different contributions in Eq. (2) are due to the vanishing of triangles, rectangles, and pentagons, respectively. Let us derive the expressions for $c_{\pm}^{\rm det}$, the corresponding probabilities in the deterministic case, for a large bubble (LB) with more than n_* sides. Later we will define the precise meaning of n_* and estimate its value.

Consider a vanishing bubble ($n \leqslant 5$) on the surface of the LB. First, let the LB be the only large neighbor of this bubble and let its other neighbors be small ($n \leqslant 7$; corresponding to the picture obtained in the numerical experiments [16,17]). We assume [20] that in this case the LB can lose a side only if there is no other possibility, i.e., only when the vanishing bubble is a triangle [6] (and not in the case of a rectangle or a pentagon). Consequently, Eq. (2) implies $c_-^{\rm det} = C_- \equiv 3w_3x_3$. At the same time, the LB will always gain one side when a pentagon vanishes. Since this case has five outcomes, the probability to gain a side in a random $T2$ process is 5 times smaller, hence $c_+^{\rm det} = C_+ \equiv 5c_+^{\rm ran}$.

In the other case, when the neighbors of the vanishing bubble include *another* large bubble (in addition to the LB), even if it is not as large as the LB, the competition between them may completely randomize the result of the $T2$ process. In this case the probabilities $c_{\pm}^{\rm det}$ should not differ from Eq. (2). Since our numerical experiments show that bubbles with $n \geqslant 8$ are large enough to randomize the $T2$ process, we introduce α, the probability for a given side of the vanishing bubble to belong to a ''randomizing'' bubble. Neglecting topological correlations we get

$$\alpha = \frac{\sum_{n=8}^{\infty} n x_n}{\sum_{n=3}^{\infty} n x_n}, \qquad (3)$$

where x_n is the topological distribution of the cluster including its boundary [15] (i.e., including all the hexagons that have at least one nonhexagonal neighbor); see Table I. We define $\Gamma_l = 1-(1-\alpha)^l$, the probability that at least one of the l sides of an $(l+1)$-sided vanishing bubble belongs to a randomizing bubble (one more side is associated with the LB). Then we can write the expressions for $c_{\pm}^{\rm det}$ in the presence of the randomizing bubbles:

$$c_+^{\rm det} = c_+^{\rm ran}\Gamma_4 + C_+(1-\Gamma_4), \qquad (4)$$

$$c_-^{\rm det} = 3w_3x_3 + 2w_4x_4\Gamma_3 + 2w_5x_5\Gamma_4. \qquad (5)$$

The first and the second terms in Eq. (4) correspond to the cases when the vanishing pentagon has and has no randomizing neighbors. Equation (5) contains three terms. Since triangles vanish without any choice, their contribution (the first term) is the same as in Eq. (2). Rectangles contribute to c_- only in the presence of a randomizing neighbor, so their contribution includes the factor Γ_3 [compare with Eq. (2)], guaranteeing that at least one of its three neighbors (other than the LB) is a randomizing one. Analogously, the pentagon's contribution includes the factor Γ_4. Finally, the transition probabilities for any β are determined by

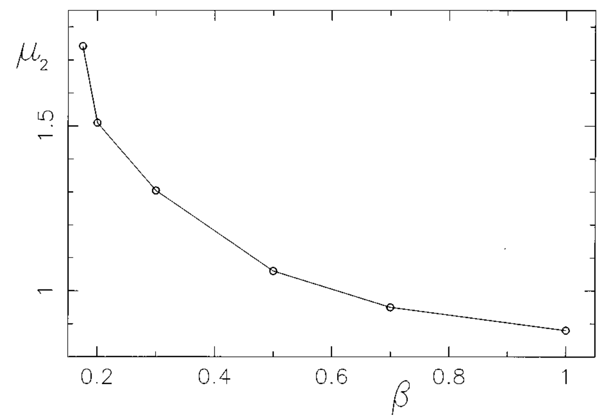

FIG. 3. Second moment of the topological distribution of the cluster μ_2 as a function of the noise parameter β for $\beta > \beta_c \approx 0.17$. μ_2 grows sharply when β approaches β_c.

$$\tilde{c}_{\pm}(\beta) = \beta c_{\pm}^{\rm ran} + (1-\beta)c_{\pm}^{\rm det}. \qquad (6)$$

Now we can explain simply the small-β behavior of the LB: if $\tilde{c}_+(\beta) > \tilde{c}_-(\beta)$, its number of sides, n will grow *linearly* with time and therefore $\mu_2 \to \infty$. This linear growth is consistent with the linear dependence of $n(\sqrt{a})$ presented in Fig. 2. Indeed, if $n \sim \sqrt{a}$, Eq. (1) implies $a \sim t^2 \to n \sim \sqrt{a} \sim t$. The condition for critical β is $\tilde{c}_+(\beta_c) = \tilde{c}_-(\beta_c)$, which gives

$$\beta_c = \delta c^{\rm det}/(\delta c^{\rm det} + \delta c^{\rm ran}), \qquad (7)$$

where $\delta c \equiv c_- - c_+$. Substituting into Eq. (3) the values for x_n obtained in our simulations (see Table I) for $\beta = 0.17$ we get $\alpha \approx 0.10$. The experimental values [21] for the rates w_n are $w_3 = 49.0$, $w_4 = 7.7$, and $w_5 = 1.0$. Using all these numbers in Eqs. (2), (4) and (5), we find $c_+^{\rm det} = 0.93$, $c_-^{\rm det} = 0.63$, $c_+^{\rm ran} = 0.26$, and $c_-^{\rm ran} = 1.48$. Then, Eq. (7) yields $\beta_c \approx 0.2$, surprisingly close to the result of our simulations, verifying our assumptions.

Thus, for $\beta < \beta_c$ any bubble with more than a critical number of sides n_* (a LB) will increase its number of sides, leading to unbounded growth of μ_2. Since n_* is finite (from our simulations we estimate it as $n_* \sim 14$), a LB unavoidably appears, as the cluster becomes larger. The LB-nucleation probability is rather small, so it may take a long time for a nucleus to appear [22]. Moreover, the fluctuations of this nucleation time are also large. Therefore, for short runs the nucleus may not appear at all and μ_2 remains bounded, causing the above-mentioned coexistence. For infinitely long runs a sharp transition at $\beta = \beta_c \approx 0.20$ is expected.

For $\beta > \beta_c$, $\tilde{c}_+(\beta) < \tilde{c}_-(\beta)$ and the LB prefers to lose sides, so $\mu_2(t)$ stays finite as $t \to \infty$: $\mu_2(t) \to \mu_2(\beta)$. Such behavior is shown experimentally in Ref. [3(b)], Fig. 5. The dependence $\mu_2(\beta)$, obtained in our simulations, is presented in Fig. 3. We can estimate $\mu_2(\beta)$ near β_c. The singular part of $\mu_2(\beta)$ is determined by the *tail* of the distribution x_n, given [19,21] roughly by $x_n \sim \gamma^n$, where $\gamma = c_+/c_-$. Since $\gamma(\beta_c) = 1$, at $\beta \approx \beta_c$ we have $\gamma(\beta) = 1 - b(\beta - \beta_c)$, $b > 0$. Then, $x_n \sim e^{-b(\beta-\beta_c)n}$, so $\mu_2(\beta) \sim (\beta - \beta_c)^{-2}$. Although our simulations show similar behavior, quantitative agreement requires much larger simulations.

Experimental study and more basic simulations would be welcome. The simplest experimental test is the transient behavior of a froth started with a very ordered configuration [3]. The *peak* of $\mu_2(t)$ observed in this case is closely related to the behavior of the single cluster [13,23]. If roughening the plates and tuning the D/d ratio destroy the peak of $\mu_2(t)$, it would confirm the effect reported in this paper. Numerical experiments using a $q=\infty$ Potts model, similar to that of [16] but at finite temperature, would also be interesting. One can study whether the temperature can randomize $T2$ processes enough to reproduce this behavior. Analogously, the same question can be posed about the quenched disorder.

I am grateful to E. Domany for critical comments. This work was supported in part by grants from the Germany–Israel Science Foundation, the United States–Israel Binational Science Foundation, and by the Clore Foundation.

[1] J. Stavans Rep. Prog. Phys. **56**, 733 (1993).
[2] J. Von Neumann, in *Metal Interfaces* (ASM, Cleveland, 1952), p. 108.
[3] (a) J. A. Glazier, S. P. Gross, and J. Stavans, Phys. Rev. A **36**, 306 (1987); (b) J. Stavans and J. A. Glazier, Phys. Rev. Lett. **62**, 1318 (1989); (c) J. A. Glazier and J. Stavans, Phys. Rev. A **40**, 7398 (1989).
[4] M. Marder, Phys. Rev. A **36**, 438 (1987).
[5] J. Stavans, Phys. Rev. A **42**, 5049 (1990).
[6] When a *triangle* vanishes all its neighbors lose a side; for a rectangle, only two of them lose a side; for a pentagon, two of them lose a side and one gains a side.
[7] V. E. Fradkov, M. O. Magnasco, D. Udler, and D. Weaire, Philos. Mag. Lett. **67**, 289 (1993).
[8] D. Weaire and J. P. Kermode, Philos. Mag. B **48**, 245 (1983); T. Herdtle and H. Aref, J. Fluid Mech. **241**, 233 (1992).
[9] B. Levitan and E. Domany, Intr. J. Mod. Phys. B (to be published).
[10] C. W. J. Beenakker, Phys. Rev. A **37**, 1697 (1988).
[11] V. E. Fradkov, L. S. Shvindlerman, and D. G. Udler, Scr. Metall. **19**, 1285 (1985); Philos. Mag. Lett. **55**, 289 (1987).
[12] B. Levitan and E. Domany, Europhys. Lett. **32**, 543 (1995).
[13] B. Levitan and E. Domany, Phys. Rev. E **54**, 2766 (1996).
[14] D. Weaire, in *Proceedings of a Conference on Disorder in Condensed Matter Physics*, edited by J. A. Blackman and J. Taguena (Clarendon, Oxford, 1991).
[15] B. Levitan, Phys. Rev. Lett. **72**, 4057 (1994).
[16] Y. Jiang, J. C. M. Mombach, and J. Glazier, Phys. Rev. E **52**, 3333 (1995).
[17] H. J. Ruskin and Y. Feng, J. Phys. Condens. Matter **7**, L553 (1995).
[18] μ_2 continues to grow very slowly for $\beta=0.20$, due to finite-size effects: when the cluster becomes large it starts to interact with itself due to the periodic boundary conditions.
[19] D. Segel, D. Mukamel, O. Krichevsky, and J. Stavans, Phys. Rev. E **47**, 812 (1993).
[20] This result comes from a detailed analysis of model C presented in Ref. [9].
[21] J. Stavans, E. Domany, and D. Mukamel, Europhys. Lett. **15**, 479 (1991).
[22] For example, at $\beta=0.17$, $x_{14}=0.000\,75$, so the first LB can appear only when the cluster contains $N\approx 1300$ bubbles, while the maximal value is $N\approx 4200$.
[23] D. Weaire, Phys. Rev. Lett. **74**, 3710 (1995).